STUDY GUIDE

Zeroing in on Number and Operations

Key Ideas and Common Misconceptions

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Our students' development of number sense and computation is essential and far more complex than we often realize. It takes considerable time for students to develop the conceptual underpinnings of this mathematical strand and to practice associated skills. Success requires that teachers fully comprehend the key ideas and common misconceptions associated with numbers and operations across the grade spans.

These four flipcharts are designed to support teachers' thinking about this critical content. Because the modules in the charts focus on key ideas, they can be used to support any curriculum or textbook series. They can help teachers plan lessons and follow up on concerns that arise. They also can be integrated during a lesson, just when their need is recognized. Most important, they help teachers follow the development of ideas from grades one through eight, to make sure that representations and language in one grade support work at the next level.

This study guide is designed to help teachers and administrators become familiar with the ideas in the charts as well as to provide questions for further deliberation and reflection. Six focus areas are suggested. They can be considered in one to six sessions, depending on the length of the sessions and the interest of the group.

Focus: An Overview of Number Sense

Introduction

These activities are designed to help you reflect on number sense. Most participants prefer to work in small groups during each section (Discuss, Investigate, and Connect) and then share a few comments with everyone before moving to the next section.

Discuss

- Describe an incident in your life in which your number sense made a difference in how things turned out (positive or negative) for you.
- Think of a current or past student whom you would identify as having good number sense. Describe two or three specific student behaviors or abilities that support this conclusion. Next, do the same for a student whom you would describe as lacking number sense.

- How do these descriptions inform your understanding of number sense?

Investigate

- Read "Sensible Number Problems" in *Grades 1–2*. Note the way the teacher introduces the activity and how multiple levels of readiness are supported. Then try the stories in the *What Makes Sense?* reproducible on page A9 of the appendix. These stories are designed to stimulate discussion; we find that many adults prefer to work in pairs as they complete them.
- Examine similar ideas in the "Making Sense with Numbers" module in *Grades 3–4*.
- Read "Millions and Billions" in *Grades 5–6*, stopping at the end of the questions screened in purple. Without doing any calculations or having any conversations, record your intuitive responses to these questions. Discuss your choices with a peer, changing answers if either of you is convinced to do so. Next, use a calculator to determine the answers to each question. Discuss your thoughts about this activity.
- Read "Fun with Numbers" in *Grades 7–8* and try the problems in the *Playing with Numbers* reproducible on page A5. Note how familiarity with specific sets of numbers is part of number sense at the upper grades.

Connect

- Generate a list of questions for your students to investigate that would give them the opportunity to think about the size of numbers in their lives.
- Identify ways to increase the integration of numerical data with other content areas at your grade level.
- Think about your students' interests. Design a bulletin board that would draw students' attention to numbers in meaningful contexts.

Focus: Numeration

Introduction

Counting is a skill we use throughout our lifetimes. Students should continue to develop their understanding of numbers and numeration all the way through the pre-secondary curriculum. It is vital that they make connections among a variety of representations of numbers. Such connections allow students to compute flexibly and to develop estimation strategies that involve translating from one representation to another. For example, to estimate 0.527 x 48, we might think $\frac{1}{2}$ of 48 = 24.

Discuss

- When do you count objects, one at a time, in your daily life? When do you find yourself using your fingers to help you keep track while you count?
- Think about the number and numeration curriculum at your grade level. Which concepts seem to be the most challenging for students? What common errors or misconceptions do your students tend to have year after year?
- Compare and contrast the descriptions of students at various grade levels. What issues can you trace over time?

Investigate

- Read the "Connecting Representations" module in *Grades 1–2* and "Connecting Representations of Numbers" in *Grades 3–4*, along with their associated reproducibles. Note how much growth is expected during these four years. Try the problems in the *What Do I Have?* reproducible on page A3 in *Grades 3–4*. Create some similar problems to challenge one another. You do not need to limit yourself to whole numbers.
- Read the **In the Classroom** section of the *Grades 3-4* module "In Order," and then play the games for which directions are given. What ideas about numbers are reinforced by playing these games?
- Consider the role of representations for decimal numbers in the *Grades 5–6* modules "Making Sense with Thousandths" and "Converting Fractions to Decimals." Discuss the

models provided; talk about which ones you find most helpful and most challenging, given your own understanding of decimals.

Connect

- Note the visual image shown in the **Meeting Individual Needs** section of the "Special Numbers" module in *Grades 7–8*. What number-related terms or facts do your students find challenging? What memory devices have your students created to help them?
- Think about the representations you use for numbers with your students. How could you increase the likelihood that your students make connections among multiple representations?
- Consider ways you could incorporate the use of a conjecture board in your classroom next week.
- How often do your students support their mathematical learning through games? How might teachers at your grade level cooperate to create a game library that would provide students with borrowing privileges on weekends?

Focus: Addition and Subtraction

Introduction

Too often, when we think about teaching addition and subtraction, we focus on the use of paper-and-pencil algorithmic skills that we learned in school. We refer to "putting down," "carrying," and "borrowing." Yet in our everyday lives most of us use alternative strategies to add or subtract instead of the procedures we were taught. We bring reasoning and sense making to our computations, and the traditional language hinders our students' abilities to make sense of their work.

Discuss

- Outside of your role as a teacher, when do you find yourself adding and subtracting?

- In which of these situations are you more likely to estimate, find an exact answer, or use mental computation?
- What language related to addition and subtraction do you think would be appropriate to use across grade levels?

Investigate

- Read how the teacher in *Grades 3–4* prepares her students to engage in "Column Addition" activities and the strategies she uses for **Meeting Individual Needs**. Play a game of *Double Out* on page A9 with one or two partners. Discuss your strategies for deciding when to end your turn.
- Read the "Number Line Integers" module in *Grades 7–8* and consider how the teacher engages her students in "walking the number line" as they model addition of integers. With a partner, play a game of *Model It* on page A6 of this book. Compare your number sentences with your partner. You may even want to model the sums or differences on a large number line on the floor, as explained in the "Number Line Integers" module.
- Read the **Potential Challenges and Misconceptions** sections of the "Join and Separate" and "Subtraction Is More Than Take-Away" modules in *Grades 1–2*. Then read the "Chip Board Integers" module in *Grades 7–8*. Compare and contrast the various models of addition and subtraction. With a partner, complete the problems in the *Model It* reproducible on page A6 in *Grades 7–8*.

Connect

- Examine word problems used at your grade level. (You don't need to limit yourself to whole numbers.) How often, even at the upper grades, are they limited to join or separate meanings of addition and subtraction? How often do the problems ask students to find answers other than the end state, after the change?
- Decide how and when you might engage your students with the *Math Wonder* activity on page A6 in the *Grades 5–6* book, or adapt it for your grade level.
- Set up a gallery walk for your class where they will post problems they have posed for the addition and subtraction equations you provide ahead of time.

Focus: Multiplication and Division

Introduction

Multiplication and division present challenges for many students and teachers. Students can no longer count quickly to compensate for lack of fact strategies and knowledge. Traditional algorithms include many steps that are rarely modeled or understood, and remainders are often interpreted incorrectly. It is essential that we help students make sense of these operations. Students respond well to representing multiplication and division with tiles or on grid paper, and such models support performing these operations with binomials in grade seven or eight.

Discuss

- What can you remember about your own experiences learning the basic multiplication and division facts? Find similarities and differences among these experiences.
- Many of us have what we can think of as a tape recorder playing in our heads when we perform a traditional algorithm. Share your "tapes" that play when you find 4119 ÷ 71 the way you were taught to do so. Compare your procedures and language. Then ask "why?" after each step and talk about the value of sense making.
- What similarities do you find in the algorithms for addition and subtraction? How do your multiplication and division algorithms relate to one another?

Investigate

- Read the "Meaning of Multiplication," "Properties of Multiplication," and "Two-Digit Multipliers" modules in the *Grades 3–4* book. Consider how models and sense making at the introductory stages can support learning of multiplication with greater numbers.
- Read the "Division Algorithms" module in *Grades 3–4* and the "Connecting Division to Multiplication" module in *Grades 5–6*. Write a division expression and use each technique presented to find the quotient.
- Read the **Potential Challenges and Misconceptions** section of the "Remainders" module in *Grades 3–4*. Create a division word problem for each of the four decisions possible

about what to do with a remainder. Learn more about how remainder issues continue by reading the "Dealing with Remainders" module in *Grades 7–8*. Play the *Remainder Rally* game on page A21 of the *Grades 3–4* book. Learning is deepened when we ask questions that encourage higher-order thinking. What questions could you ask about this game that would help students make generalizations about remainders?

- Play the *Estimation Bingo* game on page A15 of the *Grades 5–6* book. With each turn, talk about the strategies you use to choose your numbers.

Connect

- At your grade level, think about your students and where they struggle with learning to add, subtract, multiply, or divide. What new strategies might you want to try as a group to address these issues?

- Too often, students successfully learn a strategy to multiply or divide and are told that a new technique must be learned at a later grade level, but no connections are made between the different approaches. Talk across grade levels about strategies that can be supported throughout the grades and about ways to help students' techniques become more efficient as they advance in age.

- There are students at all grade levels who need to practice their basic facts. Play the game presented on page A19 in the *Grades 3–4* book. Using the idea of needing a match to move forward, create a new version of the game that would provide your students the practice they need right now.

Focus: Operating on Rational Numbers

Introduction

Ask teachers to identify the hardest area of mathematics to teach, and the majority will reply "fractions!" Fractions are difficult in part because they actually represent three things: a number, a ratio, and division. Unfortunately, many students experience fractions only as parts and wholes and have no idea where they appear on the number line.

The flipcharts address the potential challenges and misconceptions that students exhibit across all grade levels. They include a variety of activities designed to engage students in reasoning about fractions, equivalent fractions, simplifying fractions, and computing with fractions. When teachers use appropriate mathematical language and representations, there can be a significant reduction in the misconceptions students develop about fractions.

A wide variety of knowledge about fractions can be found among teachers, depending on the grade level they teach and the way they have been taught. As you discuss these ideas, encourage those who feel the least confident to speak first; others can build on the initial ideas.

Discuss

- Think of three-fourths. What is the first visual image you get in your mind's eye? Draw it, and then write words to describe it. As a group, discuss the images and words, and the implications for what students learn about fractions. Work together to create different visual images of three-fourths. Does the language change for these images? Do these images and words also work for understanding five-fourths? For answering questions related to how many and how much?

- When do you use fractions and decimals in your life? Do you have a preference for one or the other? Do you operate with fractions or convert to decimals and use a calculator? How might you estimate 0.345 x 18? What does this suggest about the need to understand operations with fractions?

- Consider the expression $2\frac{1}{2} \div \frac{1}{4}$. Discuss the meaning of this expression, and create a word problem that could be answered by finding the quotient. How were you taught to divide fractions? Discuss what you understand about why your technique works.

Investigate

- Read the **Potential Challenges and Misconceptions** sections in the "Finding Parts and Making Wholes" and "Renaming Fractions" modules in *Grades 3–4*. Also read the "Number Lines and Benchmark Fractions" module at this grade level. Talk about the models used and the language highlighted. Investigate the tasks in the *Parts and Wholes* reproducible on page A25 of this flipchart. Reconsider related ideas at a more advanced level in the "Density of Rational Numbers" module in *Grades 7–8*.

- Read and discuss the **Potential Challenges and Misconceptions** section about decimals in the "Adding and Subtracting Decimals" module in *Grades 5–6*. Which of the activities do you think best support students' sense making about decimals? Try the *What's My Number?* problems on page A32 of the *Grades 5–6* book.

- See the "Modeling Multiplication of Fractions" and "Modeling Division of Fractions with Pattern Blocks" modules in *Grades 5–6*. How might these initial models help students make sense of these operations with fractions? Use pattern blocks to complete the tasks presented on pages A23 and A24 of this flipchart.

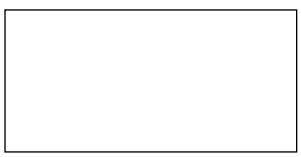
- Investigate the "Multiplying Fractions with Arrays," "Dividing Fractions with Area Model," "Multiplying Decimals with Arrays," and "Dividing Decimals" modules in *Grades 5–6*. How do these techniques support understanding across decimals and fractions? How do they compare to those used with whole numbers in *Grades 3–4*?

Connect

- Decide on the language you want to use when developing ideas about fractions across the grade levels.

- Think about the representations of fractions and decimals you use in your classrooms. What understandings do they develop best? What limitations or challenges might they support?

- Create problems in context, appropriate for your students to solve, that involve comparing fractions. For example, for grades one and two you might pose the following problem (though simple fractions are not part of the flipchart focus, students in these grades have some understanding of them):



Liam and James frosted the top of this cake. Liam frosted one-half of it, and James frosted one-fourth of it. Their mom finished the job. Who frosted more of the top of the cake, Liam or James?

Again, based on grade level, encourage students to make drawings, tell how they decided, write explanations, and show any work. Bring their responses to your next gathering, and

compare responses across classrooms and grade levels. What conclusions can you draw about student understanding?

Focus: Ratio and Proportionality

Introduction

Proportional reasoning is the main number sense focus in the middle grades. It requires students to transition from thinking additively to thinking multiplicatively. This change occurs gradually as students actively engage in meaningful tasks and problems that involve ratios and proportions represented in tables and graphs.

Just as students in the earlier grades need to think and reason through appropriate procedures for adding, subtracting, multiplying, and dividing, middle school students need to reason about ratios and proportions. Building the reasoning and sense making necessary for working with proportionality is more important than memorizing procedures.

Discuss

- Identify the ways you use proportional reasoning in the completion of everyday tasks, the enjoyment of sports and exercise, and the planning of a trip.
- Discuss the ways that proportional reasoning permeates the study of fractions, decimals, and percents, and leads to success in algebra.

Investigate

- Read the "Equivalent Values" module in *Grades 5–6*, and consider how proportional reasoning is involved in converting between fractions and decimals.
- Read the **In the Classroom** section of the "Percent of Increase or Decrease" module in *Grades 7–8*. Focus on the visual representations and organizers used to model proportional ideas. Which ones make the most sense to you? Complete the tasks in the *Percent of Change* reproducible on page A26 of the flipchart.

- Rarely are students exposed to graphing ratios. Consider how graphs are used in the "Representing Ratios on Graphs," "Working with Ratios," "Rate Tables," and "Interpreting Ratios" modules in *Grades 7–8*. Discuss how graphs can inform your—and your students'—understanding of ratios. Investigate the *Graphing Ratios* reproducible on page A29 in the *Grades 7–8* book. Think about and model how you would traditionally solve these problems before using the graphing technique.

- Too often, as mathematical ideas get more complex, hands-on experiences disappear. Read about the experiments described in the "Reasoning Proportionally" and "Solving Problems with Proportionality" modules in *Grades 7–8*. Conduct one or both experiments, and discuss opportunities you've had to experience hands-on learning as a student.

Connect

- Many students are confused by ratios. The **In the Classroom** section of "Interpreting Ratios" in the *Grades 7–8* book describes how survey data can be used to engage students in a discussion about the interpretation of ratios. Survey data is appropriate at all grade levels. What questions might you ask students about survey data?

- Many students do extremely well with ratio and proportionality when connections are made between the number and geometry strands, especially in relation to scale drawings. Brainstorm ways to make connections between these two strands that are appropriate at your grade level.