The Xs and Whys of Algebra

Key Ideas and Common Misconceptions

Anne Collins & Linda Dacey
THE Xs AND WHYs OF ALGEBRA
Key Ideas and Common Misconceptions
Anne Collins and Linda Dacey

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Introduction

Algebra is so integral to many of today’s careers that the class is often referred to as a gatekeeper course. It is also the cornerstone on which all higher mathematics is built. The long-term advantages to successful learners are so great that Robert Moses, the founder of the Algebra Project and a noted civil rights leader, has identified the learning of algebra as a civil right. Though algebraic reasoning is introduced in the earlier grades, most students usually enter into a more formal study of algebra in grade eight.

Too often, educators pay inadequate attention to the conceptual development of algebraic ideas as they focus almost exclusively on procedural knowledge. In fact, many teachers believe that algebra is simply the manipulation of variables and symbols, and that mastery of that manipulation is the goal for a successful algebra program. Further, there is a vast discrepancy among teachers about how algebra should be taught, how it relates to arithmetic, and how it connects to real-world experiences. It is no wonder, then, that teachers are often unable to identify algebra’s key ideas or address students’ common misconceptions.

By the end of grade eight, all students should have a strong foundation for algebra and should be able to reason about and make sense of algebra. This reasoning and sense making is essential to students’ future success in mathematics. This flipchart will focus on the following key ideas:

- using variables meaningfully
- using multiple representations for expressions
- connecting algebra with number
- connecting algebra with geometry
- manipulating symbols with understanding

The thirty modules in this flipchart are designed to engage all students in mathematical learning that develops conceptual understanding, addresses common misconceptions, and builds key ideas essential to future learning. The modules are research based and can be used to support response to intervention (RTI), a philosophy that utilizes quality interventions matched to student needs. They offer suggestions and resources for teachers seeking material for students identified as most likely to benefit from tier 1 or 2 supports as well as enrichment activities and challenges for all students.

Following the recommendations of the National Council of Teachers of Mathematics (2010) and the National Governors Association along with the Council of Chief State School Officers (2010), we have organized the modules at this level into three sections: Expressions, Equations, and Functions. Each module begins with the identification of its Mathematical Focus and the Potential Challenges and Misconceptions associated with those ideas. In the Classroom then suggests instructional strategies and specific activities to implement with your students. Meeting Individual Needs offers ideas for adjusting the activities to reach a broader range of learners. All modules are supported by one or more reproducibles (located in the appendix), and References/Further Reading provides resources for enriching your knowledge of the topic and gathering more ideas.

We encourage you to keep this resource on your desk or next to your plan book so that you will have these ideas at your fingertips throughout the year.

REFERENCES/FURTHER READING


Mathematical Focus

- Represent systems of inequalities on the Cartesian coordinate plane.
- Solve problems using inequalities.

Potential Challenges and Misconceptions

“Students who think a variable must represent only one value find it extremely difficult to grasp that it has the ability to cover an (un)limited range of numbers. This becomes a major barrier to students’ interpretation of the possible solutions to an inequality” (Blanco and Garrote 2007, 227). It is often difficult for students to represent, understand, and use the specific mathematical language of inequalities and to illustrate them on a graph.

In the Classroom

Inequality problems that involve a relationship between two variables require graphing on the Cartesian coordinate plane. One teacher starts with a problem that has a familiar context.

Claire mixed a batch of hot chocolate with a ratio of two parts chocolate to three parts milk. Her friends said it was too sweet. Claire then mixed a second batch with a ratio of three parts chocolate and five parts milk. Her friends told her it was not chocolatey enough. What are some chocolate-to-milk ratios she can use to make a better-tasting batch of hot chocolate?

As the teacher walks around listening to students discuss the problem, she hears Devon state, “A number line won’t work, so let’s use a graph.”

She pauses to ask Devon a clarifying question. “What criteria do you need to draw a graph?”

She also hears Jamal ask, “If we draw a ray, we can’t use open circles; what do you think shows the values come close to but don’t equal a number?” She listens to Charlie ask, “Do you think we color in the space between the rays or leave it blank?”

After an allotted period of time, Charlie’s group shares its work (see figure).

Charlie explains, “We knew the ratio was greater than three-fifths but less than two-thirds, so we decided to shade in the space between the two ratios. The only thing we could think of was to use dashed lines to show the amounts come close to but don’t equal those ratios.” The class discusses the suggested representation before examining the next problem.

Ross and Max are planting a garden of pumpkins and watermelons. They have room for no more than 60 plants. They plan on planting more pumpkins than watermelons. How many of each plant might they plant?

Many students find this problem more challenging than the previous one since they have to solve the system of inequalities before graphing it. One teacher engages his students in a facilitated solution process to ensure that all students understand this important concept. He first instructs the students to read the entire problem and discuss it until he is satisfied that the students know what the problem is asking. He then instructs the students to write a symbolic representation for the data in the problem. After a very short period of time, he asks a volunteer to write the inequalities on the board. He asks if anyone did it differently. He asks a volunteer to write the solution on the board. She writes $p \leq 60 - w$ and $p > w$.

The class discusses it, agreeing that it is correct. Next, groups graph the solution on easel-size paper. When they’ve finished, students hang their graphs on the board. The teacher challenges the class to examine all the graphs and decide which is the most accurate and why. After much discussion, the class agrees that the graph in the following figure is correct.

Meeting Individual Needs

To ensure that all students understand the symbols used on both the number line and the Cartesian coordinate plane, assign pairs of students to complete the Matching Inequalities reproducible on page A33 in the appendix. After cutting out the cards and mixing them up, each pair of students must match the cards with written inequalities to the cards with symbolic and graphical representations. Students should take turns sharing their matches, defending their selections as necessary.

REFERENCE/FURTHER READING

Mathematical Focus
- Pose problems that include number line inequalities.
- Pose problems that include Cartesian coordinate plane inequalities.

Potential Challenges and Misconceptions
When teachers give students mathematical expressions, equations, graphs, or tables, they often instruct the class to compute, simplify, solve, or graph information presented without any context. Rarely do they ask students to pose questions about the data or representations. This lack of experience dampens students’ curiosity and makes it challenging for them to identify appropriate questions or problems to research. “When students begin posing their own original mathematical questions and see these questions become the focus of discussion, their perception of the subject is profoundly altered. When they get to spend time working on these questions, their ownership of the experience produces excitement and motivation” (Cuoco, Manes, and Dash 2003). This is especially true when inequalities are involved.

In the Classroom
One teacher begins a lesson on problem posing by stating, “I need you to help me make sense of this open number line. What situations might the original question represent?”

Students work in small groups to identify appropriate contexts. Isabel’s group offers this idea:

If we have fifty girls in our eighth-grade class and our total enrollment is more than 75 but no more than 100 students, how many boys might be in the class?

The teacher shows another representation to her students, which indicates a relationship between two variables. She challenges them to pose problems that might be represented by the following graph.

Max’s group suggests a problem similar to ones the class has studied before:

What mixtures of hot chocolate to milk will be richer than two parts hot chocolate to three parts milk but no richer than the mixture of four parts hot chocolate to five parts milk?

Sabrina’s group reports next:

Levi is challenging Brandi and Brett to a walking race. He claims he walks at a rate that is faster than Brandi’s and less than Brett’s. If Brandi walks two meters in three seconds and Brett no faster than four meters in five seconds, at what rate might Levi walk?

The most challenging problem-posing situations are prompted by the following two representations.

Students need assistance with identifying potential situations for these compound inequalities. The teacher suggests the students brainstorm about times when exceptions are made for people of different ages. After an allotted period of time, students share the following circumstances: at the movies you pay less if you are under twelve or over sixty-five; on a bus you pay less fare if you are under six or over sixty-five; at an amusement park children pay less than adults and senior citizens. After the brainstorming exercise, the students pose a variety of different problems. For the figure on the left, one group suggests a question about movie ticket prices:

The Show Me Cinema charges $10.50 to see a movie. They have reduced rates for senior citizens and children. At what ages does the Show Me Cinema discount their rates?

The group writes the inequality as $y \geq 65$ or $y \leq 6$.

For the figure on the right, another group proposes this problem:

Jill’s favorite cookies contain 7 grams of fat. Jill usually eats more than three small cookies and usually fewer than eight small cookies. How many grams of fat might Jill consume?

The group represents the inequality as $21 < f < 56$.

The Problem-Posing Representations reproducible on page A34 in the appendix provides inequalities in various forms for students to write story problems about.

Meeting Individual Needs
Putting inequalities into contextual situations is most beneficial for students who struggle with using algorithms for the sake of algorithms. We have found that when our students represent situations that they confront in their own lives, they are better able to make sense of the algebraic representations and reason through them. Some students need help in identifying when those situations occur. Most benefit from brainstorming ideas.

REFERENCE/FURTHER READING
“This is an excellent resource for teachers of algebra, grounded in NCTM’s Principles and Standards for School Mathematics and organized around the Expressions, Equations, and Functions clusters of the Common Core State Standards. Each ‘Flip’ includes specific teaching challenges, points out student learning misconceptions, and contains suggestions for classroom implementation and meeting diverse student needs.”

— Mike Shaughnessy, president, NCTM

In many ways, algebra can be as challenging for teachers as it is for students. With so much emphasis placed on procedural knowledge and the manipulations of variables and symbols, it can be easy to lose sight of the key ideas that underlie algebraic thinking and the relevance algebra has to the real world. In The Xs and Whys of Algebra: Key Ideas and Common Misconceptions, Anne Collins and Linda Dacey provide a set of thirty research-based modules designed to engage all students in mathematical learning that develops conceptual understanding, addresses common misconceptions, and builds key ideas that are essential to future learning.

Designed for use in seventh- to ninth-grade courses focused on an introduction to formal algebra, this flipchart emphasizes five essential algebraic concepts: using variables meaningfully; using multiple representations for expressions; connecting algebra with number; connecting algebra with geometry; and manipulating symbols with understanding. The modules can be used to support Response to Intervention (RTI) and include resources for tier 1 or 2 supports as well as enrichment activities and challenges for all students.

Following the recommendations of the National Council of Teachers of Mathematics and the Common Core State Standards, the modules at this level are organized into three sections: Expressions, Equations, and Functions. Each module begins with the identification of its Mathematical Focus and the Potential Challenges and Misconceptions associated with those ideas. In the Classroom then suggests instructional strategies and specific activities to implement with students. Meeting Individual Needs offers ideas for adjusting the activities to reach a broader range of learners. All modules are supported by one or more reproducibles (located in the appendix), and References/Further Reading provides resources for enriching your knowledge of the topic and gathering more ideas.

Teachers will want to keep this resource next to their plan book so they will have these ideas at their fingertips throughout the year.

Grades 7–9

Anne Collins is the director of the Mathematics Program at Lesley University. She has been providing mathematics content professional development institutes and courses for teachers for the past ten years and is on the NCTM Board of Directors.

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